

KUNS-2501  
 YITP-14-46  
 KEK-TH-1740

# Emergent bubbling geometries in gauge theories with $SU(2|4)$ symmetry

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## Abstract

We study the gauge/gravity duality between bubbling geometries in type IIA supergravity and gauge theories with  $SU(2|4)$  symmetry, which consist of  $\mathcal{N} = 4$  super Yang-Mills on  $R \times S^3/Z_k$ ,  $\mathcal{N} = 8$  super Yang-Mills on  $R \times S^2$  and the plane wave matrix model. We show that the geometries are realized as field configurations in the strong coupling region of the gauge theories. On the gravity side, the bubbling geometries can be mapped to electrostatic systems with conducting disks. We derive integral equations which determine the charge densities on the disks. On the gauge theory side, we obtain a matrix integral by applying the localization to a 1/4-BPS sector of the gauge theories. The eigenvalue densities of the matrix integral turn out to satisfy the same integral equations as the charge densities on the gravity side. Thus we find that these two objects are equivalent.

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# 1 Introduction

Emergent geometry is a key concept in formulating a quantum theory of gravity because in such a theory space(time) is believed to be not fundamental but emergent. A concrete realization of this notion has been achieved by the gauge/string duality [1–3], which states an equivalence between gauge theories and string theories. For instance, the duality between the  $c = 1$  matrix model and the two-dimensional string theory is known to be a prototype of the gauge/string duality (see for reviews [4–6]). In this duality, the matrix model can be mapped to a free fermion system. The Fermi surface, which is the eigenvalue density of the  $c = 1$  matrix model, corresponds to one spatial direction of the dual two-dimensional geometry. Another well-known example is the duality between 1/2-BPS operators in  $\mathcal{N} = 4$  super Yang-Mills theory (SYM) and the bubbling geometries in type IIB supergravity [7, 8]<sup>1</sup>. The chiral primary operators in  $\mathcal{N} = 4$  SYM can also be mapped to the states of free fermions [9, 10] and its phase space can be identified with the droplets in the gravity dual [7]. The same kind of correspondence is shown to exist for the 1/2-BPS Wilson loop operators in [8]. In these cases also, the Fermi surface corresponds to one spatial direction in the dual ten-dimensional geometry. Other recent examples include the duality between Wilson surfaces in six-dimensional  $\mathcal{N} = (2, 0)$  theory and M-theory bubbling geometries [11].

Recently, yet another example of the emergent geometry has been demonstrated by the present authors and T. Okada in [12]. There, we studied the gauge/gravity duality between the plane wave matrix model (PWMM) and the bubbling geometry in type IIA supergravity [7, 13], which is explained in detail below.

PWMM is a matrix quantum mechanics originally proposed as a formulation of M-theory in the light-cone frame on the pp-wave geometry [14]. The theory is the mass deformation of the BFSS matrix theory [15] and has  $SU(2|4)$  symmetry, which consists of  $R \times SO(3) \times SO(6)$  bosonic symmetry<sup>2</sup> and 16 supersymmetries. PWMM possesses many discrete vacua, called fuzzy sphere, which are labeled by representations of  $SU(2)$ . One can make the irreducible decomposition of the representations, so that the vacua

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<sup>1</sup> The duality for less supersymmetric operators such as 1/4- and 1/8-BPS were studied, for example, in [28].

<sup>2</sup> Precisely speaking, this is the bosonic subgroup of the universal cover of  $SU(2|4)$ .

are specified by irreducible representations and their multiplicities which appear in the decomposition.

For each vacuum of PWMM, the dual gravity solution was constructed by Lin and Maldacena in type IIA supergravity, which is known as the bubbling geometry [7, 13]. If  $SU(2|4)$  symmetric ansatz is assumed, the equations of motion can be reduced to a simple differential equation with boundary conditions characterized by fermionic droplets on a particular one-dimensional line in ten-dimensions. Solving this equation is shown to be equivalent to solving for an electrostatic potential of a three-dimensional axially symmetric electrostatic system with some conducting disks [13]. In this duality, the dimensions of irreducible representations of a fuzzy sphere vacuum correspond to NS5-brane charges and their multiplicities correspond to D2-brane charges. In the associated electrostatic system, they correspond to the positions and the charges of the conducting disks. See Fig.1 for an electrostatic system for a general vacuum of PWMM.

In [12], the duality for PWMM was studied in the case of the vacuum characterized by the direct sum of copies of an irreducible representation. The gravity dual of this theory was examined in [16], where the charge density on the conducting disk in the corresponding electrostatic system was shown to satisfy a Fredholm integral equation of the second kind. In PWMM, a 1/4-BPS sector was considered. This sector is made of a complex scalar that corresponds to two spatial directions in the dual geometry on which the electrostatic system is defined. By applying the localization technique [17], this sector reduces to a matrix integral [18]. By evaluating this matrix integral, it was found that the eigenvalue density of the matrix integral obeys the same integral equation as the charge density in the electrostatic problem. This fact naturally leads us to identify the eigenvalue density with the charge density. Since the charge density completely determines the gravity solution, it was concluded that one can exactly reconstruct the gravity solution from PWMM.

Remarkably, the sector studied in [12] is interacting and so the coupling constant appears as a parameter in this duality. This is in sharp contrast to the examples given in the first paragraph, where the relevant sectors in the gauge theories are free and there is no coupling dependence. The existence of such an extra parameter makes the duality more attractive. Actually, the gauge/gravity duality for PWMM admits two interesting limits that lead to the duality for other field theories [16, 19]. One is the D2-brane limit

(commutative limit of fuzzy sphere), in which PWMM becomes SYM on  $R \times S^2$ , and the dual geometry asymptotically becomes D2-brane geometry. The other one is the NS5-brane limit, in which PWMM is considered to become type IIA little string theory (LST) [20–23] on  $R \times S^5$ , and the dual geometry asymptotically becomes NS5-brane geometry. Thus, the study of the gauge/gravity duality for PWMM enables us to study that for SYM on  $R \times S^2$  and even that for LST in a comprehensive manner.

In this paper, we extend the results of [12] to the case of a general vacuum of PWMM. We first consider the gravity side. We solve the electrostatic problem for a general vacuum of PWMM by extending the method developed in [16]. We then show that the electrostatic potential of the problem can be described in terms of the charge densities on each conducting disk and the charge densities satisfy a system of integral equations. Next, we study the gauge theory side. We consider a general vacuum and analyze the 1/4-BPS sector as in the previous paper [12]. In this case, the 1/4-BPS sector can be described by a multi-matrix integral [18]. We show that the saddle point equations for the eigenvalue densities of the matrix integral are exactly the same as the system of integral equations for the charge densities in the corresponding electrostatic problem. Thus, as in [12], we can identify the eigenvalue densities with the charge densities, which determine the gravity solution. This result shows that PWMM around a general vacuum can contain its gravity dual geometry as saddle point configurations of eigenvalues.

Moreover, we investigate the gauge/gravity duality for other gauge theories with  $SU(2|4)$  symmetry,  $\mathcal{N} = 8$  SYM on  $R \times S^2$  and  $\mathcal{N} = 4$  SYM on  $R \times S^3/Z_k$ . It was shown in [13, 24] that the gauge/gravity duality for SYM on  $R \times S^2$  can be obtained from PWMM by taking the D2-brane limit and that for SYM on  $R \times S^3/Z_k$  by taking the T-duality as well as the D2-brane limit. Using these relations, we prove the correspondence between eigenvalue densities in these theories and charge densities in the electrostatic systems associated with their gravity duals.

The organization of this paper is as follows. In section 2, we review the dual gravity solutions for gauge theories with  $SU(2|4)$  symmetry and solve general electrostatic problems associated to these solutions. In section 3, after reviewing PWMM and the matrix integral, we derive the saddle point equations for the eigenvalue densities. Then, we show that the charge densities can be identified with the eigenvalue densities. Finally,

we discuss the cases for SYM on  $R \times S^2$  and SYM on  $R \times S^3/Z_k$ .

## 2 Gravity dual of gauge theories with $SU(2|4)$ symmetry

In this section, we elaborate the Lin-Maldacena solution for a general vacuum of the gauge theories with  $SU(2|4)$  symmetry by analyzing the corresponding electrostatic system. We will show that the charge densities on each disk, which determine the gravity solution, satisfy a system of integral equations.

### 2.1 Lin-Maldacena solution

First, we review the Lin-Maldacena solution [13], which is the solution with  $SU(2|4)$  symmetry in type IIA supergravity. The solution is given by

$$\begin{aligned} ds_{10}^2 &= \left( \frac{\ddot{V} - 2\dot{V}}{-V''} \right)^{1/2} \left\{ -4 \frac{\ddot{V}}{\ddot{V} - 2\dot{V}} dt^2 - 2 \frac{V''}{\dot{V}} (dr^2 + dz^2) + 4d\Omega_5^2 + 2 \frac{V''\dot{V}}{\Delta} d\Omega_2^2 \right\}, \\ C_1 &= -\frac{(\dot{V}^2)'}{\ddot{V} - 2\dot{V}} dt, \quad C_3 = -4 \frac{\dot{V}^2 V''}{\Delta} dt \wedge d\Omega_2, \\ B_2 &= \left( \frac{(\dot{V}^2)'}{\Delta} + 2z \right) d\Omega_2, \quad e^{4\Phi} = \frac{4(\ddot{V} - 2\dot{V})^3}{-V''\dot{V}^2\Delta^2}, \end{aligned} \quad (2.1)$$

where  $\Delta = (\ddot{V} - 2\dot{V})V'' - (\dot{V}')^2$  and the dots and primes denote  $\frac{\partial}{\partial \log r}$  and  $\frac{\partial}{\partial z}$ , respectively. A remarkable feature of this solution is that it is written in terms of a single function  $V(r, z)$ . The Killing spinor equation in the supergravity imposes a condition that  $V(r, z)$  satisfies the Laplace equation in a three-dimensional axially symmetric electrostatic system, where  $r$  and  $z$  represent coordinates for the transverse and the axial directions, respectively. The regularity of the metric requires that the electrostatic system must consist of some conducting disks with radii tuned such that the charge densities vanish at the edges. In addition, from the positivity of the metric, there must be a certain background potential. So, the potential  $V(r, z)$  consists of these two contributions:

$$V(r, z) = V_{\text{b.g.}}(r, z) + \tilde{V}(r, z). \quad (2.2)$$

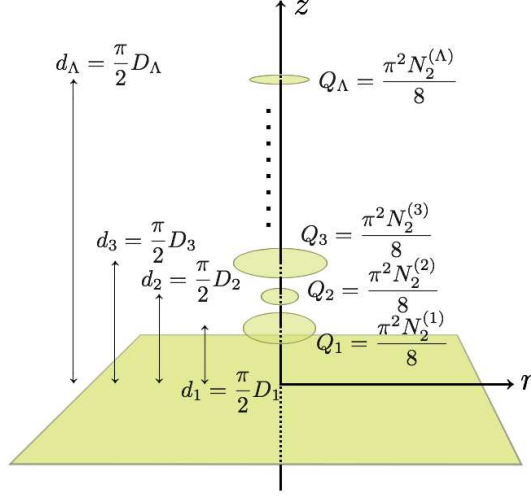


Figure 1: Electrostatic system corresponding to a general vacuum of PWMM.

The electrostatic system is determined once a theory and its vacuum are specified. The electrostatic system relevant to PWMM consists of an infinite conducting plate at  $z = 0$ , some finite conducting disks in the region of  $z \geq 0$  (Fig.1) and the background potential of the form

$$V_{\text{b.g.}}(r, z) = V_0 \left( r^2 z - \frac{2}{3} z^3 \right), \quad (2.3)$$

where  $V_0$  is a constant. The electrostatic system relevant to  $\mathcal{N} = 8$  SYM on  $R \times S^2$  consists of some finite conducting disks in the region  $-\infty \leq z \leq \infty$  (Fig.2) and the background potential of the form

$$V_{\text{b.g.}}(r, z) = W_0(r^2 - 2z^2), \quad (2.4)$$

where  $W_0$  is a constant. The electrostatic system relevant to  $\mathcal{N} = 4$  SYM on  $R \times S^3/Z_k$  consists of an infinite number of finite conducting disks arranged periodically along the  $z$ -axis (Fig.3) and the background potential (2.4)<sup>3</sup>.

The condition that the charge densities vanish at the edges of the disks relates the radii of the disks and the charges. So the independent parameters of this solution are the total charges and the  $z$ -coordinates of the disks, as well as  $V_0$  in (2.3) or  $W_0$  in (2.4).

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<sup>3</sup> Note that (2.4) is periodic up to terms of linear and constant in  $z$ , which do not contribute to the gravity solution (2.1).

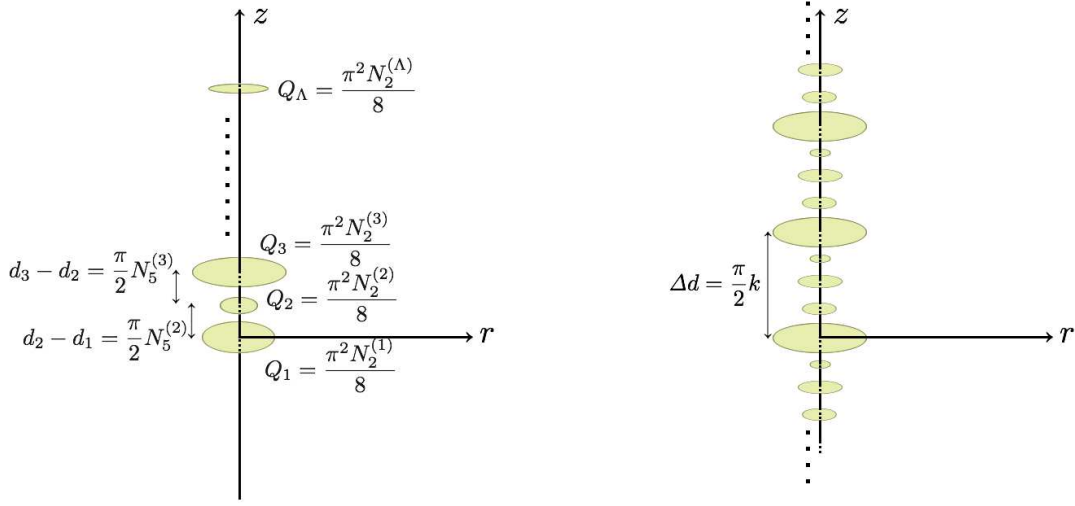


Figure 2: Electrostatic systems corresponding to a general vacuum of SYM on  $R \times S^2$  (left) and SYM on  $R \times S^3/Z_k$  (right). The conducting disks in the right figure are arranged periodically.

While  $V_0$  and  $W_0$  are related to the magnitude of the dilaton, the other parameters turn out to be related to the brane charges. This geometry has an  $S^2$  and an  $S^5$  at each point on the  $r$ - $z$  plane. One can show that on the  $z$ -axis the radius of  $S^5$  becomes zero, while on the finite conducting disks, the radius of  $S^2$  becomes zero. So one can consider a various non-contractible  $S^3$ 's or  $S^6$ 's which are given by fibering the  $S^2$  or  $S^5$  on the path (on the  $r$ - $z$  plane) ending on different disks or on different segments of the  $z$ -axis, respectively. On each  $S^3$  or  $S^6$ , one can measure the NSNS-flux or the RR-flux. This relates the parameters of the electrostatic system to the brane charges. As an example, let us consider the electrostatic system for PWMM with  $\Lambda$  finite plates (see Fig. 1). We denote the total charge, the radius and the  $z$ -coordinate of  $s$ -th disk by  $Q_s$ ,  $R_s$  and  $d_s$ , respectively, where  $s = 1, \dots, \Lambda$ . In this case, there are  $\Lambda$  independent non-contractible  $S^3$ 's and the same number of  $S^6$ 's in the geometry.  $Q_s$  and  $d_s$  are related to the D2-brane charges  $N_2^{(s)}$  and the NS5-brane charges  $N_5^{(s)}$  as

$$Q_s = \frac{\pi^2 N_2^{(s)}}{8}, \quad d_s - d_{s-1} = \frac{\pi N_5^{(s)}}{2}, \quad (2.5)$$

for  $s = 1, \dots, \Lambda$ . Here  $d_0 = 0$  denotes the position of the infinite plate.

We will show in section 3 how the parameters of the electrostatic system are mapped

to those of the corresponding gauge theory.

## 2.2 Electrostatic problem for PWMM

In the following, we derive a Fredholm integral equation which determines  $\tilde{V}(r, z)$  in (2.2). We consider the situation shown in Fig.1 and denote the solution of  $\tilde{V}(r, z)$  in the region  $z \in [d_s, d_{s+1})$  by  $V_s(r, z)$ , where  $s$  runs from 0 to  $\Lambda$  and  $(d_0, d_{\Lambda+1}) = (0, \infty)$  is assumed.

We first note that a general solution to the Laplace equation in an axially symmetric system is given by  $J_0(ru)e^{\pm zu}$  where  $J_0(z)$  is the Bessel function of the first kind of order zero and  $u$  is a positive real number. So we can write  $V_s(r, z)$  as

$$V_s(r, z) = \int_0^\infty du (C_s(u)e^{zu} + D_s(u)e^{-zu})J_0(ru). \quad (2.6)$$

Now we have the boundary condition that  $\tilde{V}(r, z) \rightarrow 0$  as  $z \rightarrow 0$  and  $z \rightarrow \infty$ . This means

$$C_0(u) = -D_0(u), \quad C_\Lambda(u) = 0. \quad (2.7)$$

We also have some continuation conditions for  $V_s$ 's at  $z = d_s$ . First, at  $z = d_s$ ,  $V_s$  should be equal to  $V_{s-1}$ . This is satisfied if

$$C_s(u)e^{d_s u} + D_s(u)e^{-d_s u} = C_{s-1}(u)e^{d_s u} + D_{s-1}(u)e^{-d_s u}. \quad (2.8)$$

Second, when  $z = d_s$  and  $r > R_s$ , not only  $V_s$  but also  $\frac{\partial V_s}{\partial z}$  should be continuous. So we have

$$\int_0^\infty du u (C_s(u)e^{d_s u} - D_s(u)e^{-d_s u} - C_{s-1}(u)e^{d_s u} + D_{s-1}(u)e^{-d_s u}) J_0(ru) = 0 \quad (2.9)$$

for  $r > R_s$ . Third, when  $z = d_s$  and  $r \leq R_s$  (i.e. on the conducting disk), the value of  $V(r, z)$  should be constant:  $V(r, d_s) = \Delta_s$ . In terms of  $C_s(u)$  and  $D_s(u)$ , this is written as

$$\int_0^\infty du (C_s(u)e^{d_s u} + D_s(u)e^{-d_s u}) J_0(ru) = \Delta_s - V_{\text{b.g.}}(r, d_s). \quad (2.10)$$

In order to solve the conditions (2.8), (2.9) and (2.10), we define

$$A_s(u) = u(C_s(u) - C_{s-1}(u))e^{d_s u} - u(D_s(u) - D_{s-1}(u))e^{-d_s u} \quad (2.11)$$



for  $s = 1, 2, \dots, \Lambda$ . From (2.7) and (2.8),  $C_s(u)$  and  $D_s(u)$  can be written in terms of  $A_s(u)$  as

$$\begin{aligned} C_s(u) &= - \sum_{t=s+1}^{\Lambda} \frac{e^{-d_t u}}{2u} A_t(u), \\ D_s(u) &= \sum_{t=1}^{\Lambda} \frac{e^{-d_t u}}{2u} A_t(u) - \sum_{t=1}^s \frac{e^{d_t u}}{2u} A_t(u). \end{aligned} \quad (2.12)$$

By substituting (2.11) and (2.12) to (2.9) and (2.10), we obtain

$$\begin{aligned} \int_0^{\infty} u^{-1} \sum_{t=1}^{\Lambda} (\delta_{st} + k_{st}(u)) A_t(u) J_0(ru) du &= F_s(r), \quad (0 \leq r \leq R_s) \\ \int_0^{\infty} A_s(u) J_0(ru) du &= 0, \quad (R_s \leq r) \end{aligned} \quad (2.13)$$

where  $k_{st}(u)$  and  $F_s(r)$  are given by

$$\begin{aligned} k_{st}(u) &= -e^{-(d_s+d_t)u} + (1 - \delta_{st})e^{-|d_s-d_t|u}, \\ F_s(r) &= -2(\Delta_s - V_{b.g.}(r, d_s)). \end{aligned} \quad (2.14)$$

As shown in appendix A, the equations (2.13) can be reduced to the integral equations, (A.28) and (A.29), for the functions  $h_s(u)$  defined by (A.27)<sup>4</sup>. For our problem, it is more convenient to work with the variables

$$f_s(u) = -\frac{1}{4\sqrt{\pi}} u h_s(u). \quad (2.15)$$

Then, (A.29) is written for  $\{f_s(x)\}$  as

$$\begin{aligned} f_s(x) + \frac{1}{\pi} \sum_{t=1}^{\Lambda} \int_{-R_t}^{R_t} du \left[ -\frac{d_s + d_t}{(d_s + d_t)^2 + (x - u)^2} + \frac{|d_s - d_t|}{(d_s - d_t)^2 + (x - u)^2} \right] f_t(u) \\ = \frac{1}{\pi} \left( \Delta_s + \frac{2}{3} V_0 d_s^3 - 2V_0 d_s x^2 \right), \end{aligned} \quad (2.16)$$

and (A.28) shows that  $f_s$  is vanishing outside the region  $[-R_s, R_s]$ . Here we have defined  $f_s(x)$  with negative  $x$  as  $f_s(x) = f_s(-x)$  and extended the domain to the entire real line.

The function  $f_s(x)$  can be interpreted as the charge density on the  $s$ -th conducting disk as follows. For  $z = d_s$  and  $r \leq R_s$ , we have

$$\frac{\partial}{\partial z} V_s(r, d_s) - \frac{\partial}{\partial z} V_{s-1}(r, d_s) = K_{0, -\frac{1}{2}} h_s(r) = 4 \int_r^{R_s} \frac{f'_s(u)}{\sqrt{u^2 - r^2}} du. \quad (2.17)$$

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<sup>4</sup>  $n$ ,  $I_{s1}$  and  $I_{s2}$  in appendix A corresponds to  $\Lambda$ ,  $[0, R_s)$  and  $[R_s, \infty)$  in our problem, respectively.

On the other hand, this is equal to  $-4\pi\sigma(r)$ , where  $\sigma(r)$  is the charge density for the  $r$  direction. Hence, the total charge on the disk can be computed as

$$Q_s = -2 \int_0^{R_s} dr \int_r^{R_s} du \frac{r f'_s(u)}{\sqrt{u^2 - r^2}} = \int_{-R_s}^{R_s} du f_s(u). \quad (2.18)$$

These relations show that  $f_s(u)$  corresponds to the charge density on the  $s$ -th plate projected onto a diameter direction. These densities are fully determined by (2.16) and so is the potential which can be written in terms of  $\{f_s(u)\}$  as

$$\tilde{V}(r, z) = \sum_{s=1}^{\Lambda} \int_{-R_s}^{R_s} dt \left[ \frac{1}{\sqrt{(z - d_s + it)^2 + r^2}} - \frac{1}{\sqrt{(z + d_s + it)^2 + r^2}} \right] f_s(t). \quad (2.19)$$

Note that  $R_s$  and  $\Delta_s$  are determined by  $f_s(R_s) = 0$  and (2.18).

### 2.3 Electrostatic problem for $\mathcal{N} = 8$ SYM on $R \times S^2$

The electrostatic system associated with the gravity dual of  $\mathcal{N} = 8$  SYM on  $R \times S^2$  is shown in Fig.2 (left). The case where  $\Lambda = 2$  and  $R_1 = R_2$  was studied in [26]. Here, we generalize their result. It was shown in [16, 24] that the solution for this system can be obtained from the solution for PWMM by taking the D2-brane limit. After the redefinitions  $d_s \rightarrow d + d_s$  ( $1 \leq s \leq \Lambda$ ),  $z \rightarrow d + z$ , D2-brane limit is written as

$$d \rightarrow \infty, \quad Q_s : \text{fixed}, \quad V_0 d = W_0 : \text{fixed}. \quad (2.20)$$

Indeed, in this limit, Fig.1 becomes Fig.2 (left) and the background potential for PWMM (2.3) becomes

$$V_{\text{b.g.}}(r, z) \rightarrow -W_0 \left( \frac{2d^2}{3} + 2dz \right) + W_0(r^2 - 2z^2). \quad (2.21)$$

One can neglect the first term since it does not affect the gravity solution which depends only on  $\dot{V}$ ,  $\ddot{V}$ ,  $\dot{V}'$  and  $V''$ . Thus, the background potential for PWMM (2.3) exactly reduces to that for SYM on  $R \times S^2$  (2.4) in the limit (2.20).

By taking the D2-brane limit (2.20) of the integral equation (2.16) and the potential (2.19), we obtain

$$f_s(x) + \frac{1}{\pi} \sum_{t=1}^{\Lambda} \int_{-R_t}^{R_t} du \frac{|d_s - d_t|}{(d_s - d_t)^2 + (x - u)^2} f_t(u) = \frac{1}{\pi} (\Delta'_s + 2W_0 d_s^2 - 2W_0 x^2), \quad (2.22)$$

and

$$\tilde{V}(r, z) = \sum_{s=1}^{\Lambda} \int_{-R_s}^{R_s} dt \frac{f_s(t)}{\sqrt{(z - d_s + it)^2 + r^2}}, \quad (2.23)$$

respectively. Here  $\Delta'_s$  is a constant potential on the  $s$ -th disk,  $V(r, d_s) = \Delta'_s$  ( $r < R_s$ ). Note that  $R_s$  and  $\Delta'_s$  are determined by  $f_s(R_s) = 0$  and (2.18). The solution of (2.22) gives a general solution to the electrostatic problem for SYM on  $R \times S^2$ .

## 2.4 Electrostatic problem for $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$

The electrostatic system associated with the gravity dual of  $\mathcal{N} = 4$  SYM on  $R \times S^3/Z_k$  is shown in Fig.2 (right). The case for the trivial vacuum was studied in [26]. Here, we generalize their result. This can be obtained from that for SYM on  $R \times S^2$  by compactifying the  $z$  direction to  $S^1$  with the background potential intact [24].

We start from the solution for SYM on  $R \times S^2$ , (2.22) and (2.23), with disks periodically arranged. We change the labelling of the disks so that they are labelled by two integers  $(s, \alpha)$ , where  $-\infty \leq s \leq \infty$  and  $\alpha \in K \subset \{1, 2, \dots, k\}$ .  $s$  is a label of a single period and  $\alpha$  is that of each disk in the period. So each period consists of  $|K|$  conducting disks. We put the position of each disk to be

$$d_{s,\alpha} = \frac{\pi}{2} (ks + \alpha - 1). \quad (2.24)$$

The charge  $Q_{s,\alpha}$  and the radius  $R_{s,\alpha}$  of each disk is independent of  $s$ :  $Q_{s,\alpha} = Q_\alpha$  and  $R_{s,\alpha} = R_\alpha$ . The charge density  $f_{s,\alpha}(r)$  on each disk should also be independent of  $s$ :

$$f_{s,\alpha}(r) = f_\alpha(r). \quad (2.25)$$

Note that the naive substitutions of these conditions to (2.22) and (2.23) do not make sense because of the divergences coming from the periodicity. As remarked in [26], this divergence can be avoided by solving the electrostatic problem for the electric field rather than the potential. Hence, by differentiating (2.22) with respect to  $x$  and imposing the periodicity condition, one can obtain the integral equations for the charge densities  $\{f_\alpha(r)\}$ ,

$$f'_\alpha(x) + \sum_{\beta \in K} \int_{-R_\beta}^{R_\beta} du K_k \left( \frac{\alpha - \beta}{k}, x, u \right) f'_\beta(u) = -\frac{4}{\pi} W_0 x, \quad (2.26)$$

where

$$K_k(\nu, x, u) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dp \frac{\cosh \left\{ \frac{\pi k}{2} p \left( |\nu| - \frac{1}{2} \right) \right\}}{\sinh \frac{\pi k}{4} |p|} \left( e^{ip(x-u)} - e^{ip(x+u)} \right). \quad (2.27)$$

The electric field generated by the conducting disks is obtained from (2.23) as

$$\begin{aligned} E_r &= \sum_{s=-\infty}^{\infty} \sum_{\alpha \in K} \int_{-R_\alpha}^{R_\alpha} dt \frac{r f_\alpha(t)}{((z - 2d_{s,\alpha} + it)^2 + r^2)^{\frac{3}{2}}}, \\ E_z &= \sum_{s=-\infty}^{\infty} \sum_{\alpha \in K} \int_{-R_\alpha}^{R_\alpha} dt \frac{(z - 2d_{s,\alpha} + it) f_\alpha(t)}{((z - 2d_{s,\alpha} + it)^2 + r^2)^{\frac{3}{2}}}. \end{aligned} \quad (2.28)$$

### 3 Emergent bubbling geometry in theories with $SU(2|4)$ symmetry

In this section, we investigate PWMM around a general vacuum as well as  $\mathcal{N} = 8$  SYM on  $R \times S^2$  and  $\mathcal{N} = 4$  SYM on  $R \times S^3/Z_k$ . First, we briefly review the results of PWMM obtained in [18], in which the localization was applied to a 1/4-BPS sector and a matrix integral that describes this sector was obtained. We then show that the eigenvalue densities of the matrix integral satisfy the same integral equations as the charge densities of the corresponding electrostatic system. We also show that the same relation holds for the other gauge theories. Some properties of these theories are reviewed in appendix B.

#### 3.1 Localization in PWMM

We first review PWMM. We follow the notation used in [18]. The action of PWMM is given in the ten-dimensional notation as<sup>5</sup>,

$$S = \frac{1}{g^2} \int d\tau \text{Tr} \left( \frac{1}{4} F_{MN} F^{MN} + \frac{m^2}{8} X_m X^m + \frac{i}{2} \Psi \Gamma^M D_M \Psi \right), \quad (3.1)$$

where

$$\begin{aligned} F_{1M} &= D_1 X_M = \partial_1 X_M - i[X_1, X_M] \quad (M \neq 1), \\ F_{ab} &= m \varepsilon_{abc} X_c - i[X_a, X_b], \quad F_{am} = D_a X_m = -i[X_a, X_m], \quad F_{mn} = -i[X_m, X_n], \\ D_1 \Psi &= \partial_1 \Psi - i[X_1, \Psi], \quad D_a \Psi = \frac{m}{8} \varepsilon_{abc} \Gamma^{bc} \Psi - i[X_a, \Psi], \quad D_m \Psi = -i[X_m, \Psi]. \end{aligned} \quad (3.2)$$

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<sup>5</sup> Here, the time direction is assumed to be the Euclidean signature.

The ranges of the indices are  $M, N = 1, \dots, 10$ ,  $a, b = 2, 3, 4$  and  $m, n = 5, \dots, 10$ .  $X_1$  is the one-dimensional gauge field,  $X_a$  and  $X_m$  are  $SO(3)$  and  $SO(6)$  scalar fields and  $\Psi$  is a 16 component fermionic field. In the following we put the mass parameter  $m = 2$ .

The vacuum of PWMM is given by the fuzzy sphere solution, which is a matrix representation of  $SU(2)$  algebra

$$X_a = -2L_a = \bigoplus_{s=1}^{\Lambda} \mathbf{1}_{N_2^{(s)}} \otimes L_a^{[D_s]}. \quad (3.3)$$

Any representation of  $SU(2)$  is an  $SU(2|4)$  symmetric vacuum. The right-hand side of (3.3) stands for the irreducible decomposition, where  $L_a^{[D]}$  are generators of  $SU(2)$  algebra in  $D$ -dimensional irreducible representation,  $N_2^{(s)}$  are multiplicities of  $L_a^{[D_s]}$  and  $\Lambda$  is the number of different irreducible representations.

The parameters of PWMM around the vacuum (3.3) are identified with those of the electrostatic system in the gravity dual as [13]<sup>6</sup>

$$Q_s = \frac{\pi^2 N_2^{(s)}}{8}, \quad d_s = \frac{\pi}{2} D_s = \frac{\pi}{2} \sum_{t=1}^s N_5^{(t)}. \quad (3.4)$$

In addition, in [16], the case of  $\Lambda = 1$  was considered and  $V_0$  was inferred to be related to the gauge coupling as

$$V_0 = \frac{hm^3}{8g^2}, \quad (3.5)$$

where  $h$  is a constant. In [12], the constant  $h$  was determined as

$$h = \frac{2}{\pi^2}. \quad (3.6)$$

We claim that the relation (3.5) also holds for the theory around the general vacuum. In general,  $V_0$  should be a function of  $g^2$  and  $\hat{N}$ , where  $\hat{N} = \sum_s N_2^{(s)} D_s$  is the total matrix size [16]. Since we are interested in the region where the supergravity approximation is valid,  $\hat{N}$  must be large. Now, let us consider the case where  $D_s \gg D_1$  for any  $s > 1$ . Then, we are left with the electrostatic system made of an infinite conducting plate and a single conducting disk. This is exactly the same situation studied in [16], so that  $V_0$

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<sup>6</sup> Note that  $N_5^{(s)}$  is used in different meanings in the present paper and in [12].

should be given by (3.5). Thus, if  $N_2^{(1)} D_1$  is large, the relation (3.5) holds. Since large  $D_s$  means large  $\hat{N}$ , we conclude that  $V_0(g^2, \hat{N}) = \frac{hm^3}{8g^2}$  when  $\hat{N} \gg 1$ .

Since PWMM has one noncompact direction, in order to define the theory around a fixed vacuum precisely we have to specify the boundary condition. Here we choose the boundary condition such that all fields approach to the vacuum configuration as  $\tau \rightarrow \pm\infty$ . We also consider the 't Hooft limit, in which the tunneling between vacua are suppressed. Thus, the path integral with these conditions correctly defines PWMM around the fixed vacuum.

In the following, we consider PWMM around a general fuzzy sphere vacuum given in (3.3). In this theory we focus on the complex scalar

$$\phi(\tau) = -X_4(\tau) + \sinh \tau X_9(\tau) + i \cosh \tau X_{10}(\tau). \quad (3.7)$$

$\phi$  is invariant under four supersymmetries (1/4-BPS) after the Wick-rotation along  $X_{10}$ . From a symmetry argument,  $\phi$  was found to describe the  $r, z$  directions in (2.1) [12]. So, this 1/4-BPS sector is expected to correspond to the electrostatic system on the gravity side. Applying the localization to operators made of  $\phi$ , we obtain [18]

$$\left\langle \prod_a \text{Tr} f_a(\phi(\tau_a)) \right\rangle = \left\langle \prod_a \text{Tr} f_a(2L_4 + iM) \right\rangle_{MM}. \quad (3.8)$$

Here  $M$  is a  $\tau$ -independent Hermitian matrix with the following block structure,

$$M = \bigoplus_{s=1}^{\Lambda} (M_s \otimes \mathbf{1}_{D_s}), \quad (3.9)$$

where  $M_s$  ( $s = 1, \dots, \Lambda$ ) are  $N_2^{(s)} \times N_2^{(s)}$  Hermitian matrices. In the right-hand side  $\langle \dots \rangle_{MM}$  stands for an expectation value with respect to the following partition function,

$$Z_{\mathcal{R}} = \int \prod_{s=1}^{\Lambda} \prod_{i=1}^{N_2^{(s)}} dm_{si} Z_{1\text{-loop}}(\mathcal{R}, \{m_{si}\}) e^{-\frac{2}{g^2} \sum_s \sum_i D_s m_{si}^2}, \quad (3.10)$$

where  $\mathcal{R}$  denotes the representation of (3.3),  $m_{si}$ 's are eigenvalues of  $M_s$  and

$$Z_{1\text{-loop}} = \prod_{s,t=1}^{\Lambda} \prod_J \prod_{i=1}^{N_2^{(s)}} \prod_{j=1}^{N_2^{(t)}} \left[ \frac{\{(2J+2)^2 + (m_{si} - m_{tj})^2\} \{(2J)^2 + (m_{si} - m_{tj})^2\}}{\{(2J+1)^2 + (m_{si} - m_{tj})^2\}^2} \right]^{\frac{1}{2}}. \quad (3.11)$$

In (3.11), the product of  $J$  runs from  $|D_s - D_t|/2$  to  $(D_s + D_t)/2 - 1$ .  $\prod'$  represents that the second factor in the numerator with  $s = t$ ,  $J = 0$  and  $i = j$  is not included in this product.

### 3.2 Localization in SYM on $R \times S^2$ and SYM on $R \times S^3/Z_k$

In this subsection, we show the results of the localization for SYM on  $R \times S^2$  and SYM on  $R \times S^3/Z_k$  around a general vacuum, which can be obtained by applying the D2-brane limit and the T-duality to (3.10) with (3.11) [24].

#### SYM on $R \times S^2$

SYM on  $R \times S^2$  around a general monopole vacuum (B.3) can be obtained from PWMM in the D2-brane limit (2.20). The D2-brane limit in the PWMM language reads

$$D_s = D + 2q_s \quad (1 \leq s \leq \Lambda), \quad D \rightarrow \infty, \quad N_2^{(s)} : \text{fixed}, \quad \frac{D}{g^2} = \frac{4\pi}{m^2 g_{S^2}^2} : \text{fixed}. \quad (3.12)$$

In this limit, (3.10) with (3.11) becomes, up to an overall constant,

$$\begin{aligned} Z_{\{(q_s, N_2^{(s)})\}}^{R \times S^2} &= \int \prod_{s=1}^{\Lambda} \prod_{i=1}^{N_2^{(s)}} dm_{si} \prod_{s=1}^{\Lambda} \Delta(m_s)^2 \prod_{s=1}^{\Lambda} \prod_{i,j=1}^{N_2^{(s)}} \left[ \frac{1 + \left(\frac{m_{si} - m_{sj}}{2}\right)^2}{\{1 + (m_{si} - m_{sj})^2\}^2} \right]^{\frac{1}{2}} \\ &\times \prod_{s,t=1}^{\Lambda} \prod_{\substack{J=|q_s - q_t| \\ J \neq 0}}^{\infty} \prod_{i=1}^{N_2^{(s)}} \prod_{j=1}^{N_2^{(t)}} \left[ \frac{\left\{1 + \left(\frac{m_{si} - m_{tj}}{2J+2}\right)^2\right\} \left\{1 + \left(\frac{m_{si} - m_{tj}}{2J}\right)^2\right\}}{\left\{1 + \left(\frac{m_{si} - m_{tj}}{2J+1}\right)^2\right\}^2} \right]^{\frac{1}{2}} e^{-\frac{m\pi}{g_{S^2}^2} \sum_{s,i} m_{si}^2}, \end{aligned} \quad (3.13)$$

where  $\Delta(m_s) = \prod_{i < j} (m_{si} - m_{sj})$  is the Vandermonde determinant. The parameters of SYM on  $R \times S^2$  around (B.3) correspond to those in the gravity dual as

$$Q_s = \frac{\pi^2 N_2^{(s)}}{8}, \quad d_s = \pi q_s, \quad W_0 = \frac{\pi^2 h m}{4g_{S^2}^2}. \quad (3.14)$$

#### SYM on $R \times S^3/Z_k$

SYM on  $R \times S^3/Z_k$  around a general vacuum characterized by the holonomy (B.7) can be obtained by applying the T-duality to SYM on  $R \times S^2$ . The T-duality can be realized in the

same manner as in section 2.4. In (3.13), we first replace  $s \rightarrow (s, \alpha)$  where  $-\infty \leq s \leq \infty$  and  $\alpha \in K \subset \{1, 2, \dots, k\}$ . Next, we set

$$2q_{s\alpha} = ks + \alpha - 1, \quad (3.15)$$

and then drop the  $s$ -dependence of  $m_{s\alpha i}$  and  $N_2^{(s\alpha)}$ :  $m_{s\alpha i} \rightarrow m_{\alpha i}$  and  $N_2^{(s\alpha)} \rightarrow N_2^{(\alpha)}$ . Finally, up to an overall constant, we end up with

$$\begin{aligned} Z_{\{(\alpha, N_2^{(\alpha)})\}}^{R \times S^3/Z_k} &= \int \prod_{\alpha} \prod_{i=1}^{N_2^{(\alpha)}} dm_{\alpha i} \prod_{\alpha} \Delta(m_{\alpha})^2 \prod_{\alpha} \prod_{i,j=1}^{N_2^{(\alpha)}} \left[ \frac{1 + \left(\frac{m_{\alpha i} - m_{\alpha j}}{2}\right)^2}{\{1 + (m_{\alpha i} - m_{\alpha j})^2\}^2} \right]^{\frac{1}{2}} \\ &\times \prod_{u=-\infty}^{\infty} \prod_{\alpha, \beta} \prod_{\substack{J=|ku/2+(\alpha-\beta)/2| \\ J \neq 0}}^{\infty} \prod_{i=1}^{N_2^{(\alpha)}} \prod_{j=1}^{N_2^{(\beta)}} \left[ \frac{\left\{1 + \left(\frac{m_{\alpha i} - m_{\beta j}}{2J+2}\right)^2\right\} \left\{1 + \left(\frac{m_{\alpha i} - m_{\beta j}}{2J}\right)^2\right\}}{\left\{1 + \left(\frac{m_{\alpha i} - m_{\beta j}}{2J+1}\right)^2\right\}^2} \right]^{\frac{1}{2}} \\ &\times e^{-\frac{4\pi^2}{kg_{S^3/Z_k}^2} \sum_{\alpha, i} m_{\alpha i}^2}, \end{aligned} \quad (3.16)$$

where  $\alpha, \beta$  run over the elements of  $K$ ,  $\Delta(m) = \prod_{i < j} (m_i - m_j)$  is the Vandermonde determinant. We have also used

$$\frac{1}{g_{S^2}^2} = \frac{4\pi}{mkg_{S^3/Z_k}^2}. \quad (3.17)$$

### 3.3 Correspondence to the gravity side

Now, we evaluate the matrix integral (3.10) with (3.11) in the regime where in the gravity side the classical gravity approximation is valid. This regime corresponds in the gauge theory side to the 't Hooft limit

$$N_2^{(s)} \rightarrow \infty, \quad \lambda^{(s)} = g^2 N_2^{(s)} = \text{fixed}, \quad (3.18)$$

and

$$D_s - D_{s-1} \gg 1, \quad \lambda^{(s)} \gg D_s, \quad (3.19)$$

for arbitrary  $s$ . The 't Hooft limit (3.18) suppresses the bulk string coupling while the conditions (3.19) make the  $\alpha'$  corrections negligible. Note that  $D_s - D_{s-1} = \frac{\pi}{2} N_5^{(s)}$  is the square of the  $S^5$  radius in the  $s$ -th NS5-brane throat [13]. Also, as shown in appendix C,



the second condition of (3.19) means that the radius of  $S^5$  near the tip of a disk in the electrostatic system is large. In these limits, (3.18) and (3.19), one can evaluate (3.8) by applying the saddle point approximation, which becomes exact in these limits.

When  $D_s - D_{s-1} \gg 1$ , one can rewrite the measure factor in (3.11) as

$$\begin{aligned} & \prod_{J=0}^{D_s-1} \frac{\{(2J+2)^2 + (m_{si} - m_{sj})^2\} \{(2J)^2 + (m_{si} - m_{sj})^2\}}{\{(2J+1)^2 + (m_{si} - m_{sj})^2\}^2} \\ &= \tanh^2 \frac{\pi(m_{si} - m_{sj})}{2} \exp \left\{ \frac{2D_s}{(2D_s)^2 + (m_{si} - m_{sj})^2} - \dots \right\} \end{aligned} \quad (3.20)$$

for  $s = t$ , and

$$\begin{aligned} & \prod_{J=|D_s-D_t|/2}^{(D_s+D_t)/2-1} \frac{\{(2J+2)^2 + (m_{si} - m_{tj})^2\} \{(2J)^2 + (m_{si} - m_{tj})^2\}}{\{(2J+1)^2 + (m_{si} - m_{tj})^2\}^2} \\ &= \exp \left\{ \frac{D_s + D_t}{(D_s + D_t)^2 + (m_{si} - m_{tj})^2} - \frac{|D_s - D_t|}{(D_s - D_t)^2 + (m_{si} - m_{tj})^2} + \dots \right\} \end{aligned} \quad (3.21)$$

for  $s \neq t$ , where “ $\dots$ ” stands for  $1/(D_s \pm D_t)$  corrections. We introduce the eigenvalue densities defined for each  $s$  as

$$\rho^{(s)}(x) = \sum_{i=1}^{N_2^{(s)}} \delta(x - m_{si}). \quad (3.22)$$

In the large  $N_2^{(s)}$  limit (3.19),  $\rho^{(s)}(x)$ 's become continuous functions. Then, we obtain the effective action for (3.10) with (3.11)

$$\begin{aligned} S_{eff} &= \sum_{s=1}^{\Lambda} \frac{2D_s}{g^2} \int dx \, x^2 \rho^{(s)}(x) - \sum_{s=1}^{\Lambda} \frac{1}{2} \int dx dy \, \log \tanh^2 \frac{\pi(x-y)}{2} \rho^{(s)}(x) \rho^{(s)}(y) \\ &\quad - \sum_{s,t=1}^{\Lambda} \frac{1}{2} \int dx dy \, \left[ \frac{D_s + D_t}{(D_s + D_t)^2 + (x-y)^2} - \frac{|D_s - D_t|}{(D_s - D_t)^2 + (x-y)^2} \right] \rho^{(s)}(x) \rho^{(t)}(y) \\ &\quad - \sum_{s=1}^{\Lambda} \mu_s \left( \int dx \rho^{(s)}(x) - N_2^{(s)} \right), \end{aligned} \quad (3.23)$$

where  $\mu_s$ 's are the Lagrange multipliers for the normalization of  $\rho^{(s)}(x)$ 's.

We assume that  $\rho^{(s)}(x)$  has its support on  $[-x_m^{(s)}, x_m^{(s)}]$ . As shown in appendix C, in the limit of (3.18) and (3.19) the extents of  $\rho^{(s)}(x)$  become large;  $x_m^{(s)} \gg 1$ . Using the

fact that  $x_m \log \tanh^2 \frac{\pi x_m y}{2}$  can be approximated to  $-\pi \delta(y)$  as  $x_m \rightarrow \infty$ , we obtain the following saddle point equations

$$\begin{aligned} \rho^{(s)}(x) + \frac{1}{\pi} \sum_{t=1}^{\Lambda} \int_{-x_m^{(t)}}^{x_m^{(t)}} du \left[ -\frac{D_s + D_t}{(D_s + D_t)^2 + (x - u)^2} + \frac{|D_s - D_t|}{(D_s - D_t)^2 + (x - u)^2} \right] \rho^{(t)}(u) \\ = \frac{\mu_s}{\pi} - \frac{2D_s}{\pi g^2} x^2, \end{aligned} \quad (3.24)$$

where  $x_m^{(s)}$  and  $\mu_s$  are determined from

$$\rho^{(s)}(x_m^{(s)}) = 0 \quad \text{and} \quad \int_{-x_m^{(s)}}^{x_m^{(s)}} dx \rho^{(s)}(x) = N_2^{(s)}. \quad (3.25)$$

Notice that the saddle point equations of the eigenvalue densities (3.24) take a very similar form as the integral equations for the charge densities (2.16). In fact, by using the relations (3.4) and (3.5), one can find that they are exactly the same equations. Thus, we arrive at the relations

$$g^2 \rho^{(s)}(x) = \frac{1}{V_0} \left( \frac{2}{\pi} \right)^3 f_s \left( \frac{\pi}{2} x \right), \quad (3.26)$$

and

$$\frac{\pi}{2} x_m^{(s)} = R_s. \quad (3.27)$$

Namely, the eigenvalue density on the gauge theory side has exactly the same functional form and parameter dependence as the charge density on the gravity side, up to the trivial rescaling. Hence, they can naturally be identified with each other and this identification relates the degrees of freedom on the gauge theory side to the background geometry on the gravity side. By integrating both sides of (3.26) over  $[-x_m^{(s)}, x_m^{(s)}]$  and using (2.18) and (3.25), we find that those relations are consistent with (3.5) and (3.6).

If one finds exact solutions of (2.16) and (3.24), one can check the relations, (3.26) and (3.27), more explicitly. Although we could not find general exact solutions, still we can solve those equations in particular parameter regions. If a conducting disk is isolated at a distance from the other disks, the term with an integration in the integral equation of the disk becomes negligible. Then, the solution is simply given by a quadratic function. This is effectively the same situation as the D2-brane limit with  $\Lambda = 1$  considered in [12]. In the same way, we can consider two isolated disks which effectively form the same system

as the NS5-brane limit with  $\Lambda = 1$  [12]. In these cases, one can find the exact solutions and check the relations, (3.27) and (3.26), more directly.

The equivalence between the charge density and the eigenvalue density also holds for the other gauge theories with  $SU(2|4)$  symmetry. As shown in section 3.2, in the D2-brane limit, the partition function (3.10) reduces to the matrix integral for SYM on  $R \times S^2$  given by (3.13). In addition, by taking the T-duality, we end up with the matrix integral for SYM on  $R \times S^3/Z_k$  given by (3.16). If we apply the corresponding limits to the integral equation (2.16) of the charge density, we obtain the integral equations (2.22) and (2.26) for SYM on  $R \times S^2$  and SYM on  $R \times S^3/Z_k$ , respectively. In these cases, the integral equations (2.22) and (2.26) for the charge densities can also be identified with the saddle point equations for (3.13) and (3.16), respectively, where the same relations as (3.26) and (3.27) hold.

## 4 Summary

In this paper, we showed that the bubbling geometries in type IIA supergravity are realized in the gauge theories with  $SU(2|4)$  symmetry. We found that the charge densities of the electrostatic systems in the gravity dual are equivalent to the eigenvalue densities of the matrix integrals which govern the 1/4-BPS sector of the gauge theories.

On the gravity side, the bubbling geometries are given in terms of the electrostatic potential of electrostatic systems with conducting disks. First, we have considered the electrostatic system corresponding to PWMM around a general vacuum. We have shown that the boundary conditions of the potential are given by a system of dual integral equations. Extending the method to analyze the dual integral equations written in [27], we have reduced the dual integral equations to the Fredholm integral equations of the second kind for the charge densities on the disks. By taking the D2-brane limit or performing the T-duality as well as the D2-brane limit, we have also obtained the same type of integral equations for the charge densities in the electrostatic system corresponding to SYM on  $R \times S^2$  or SYM on  $R \times S^3/Z_k$ .

On the gauge theory side, we have investigated the matrix integrals that describe 1/4-BPS sectors of the gauge theories. First, we have considered the case for PWMM around a general vacuum in the regime where the supergravity approximation is valid.

In this regime, we have derived the saddle point equations of the eigenvalue densities of the matrix integral, which are almost the same integral equations for the charge densities on the gravity side. Then we have found that under the identifications of (3.4) and (3.5) the integral equations of the eigenvalue densities are exactly equivalent to those for the charge densities. As the D2-brane limit and the T-duality of PWMM lead to the other gauge theories with  $SU(2|4)$  symmetry, that is, SYM on  $R \times S^2$  and SYM on  $R \times S^3/Z_k$ , we have also shown the equivalence of the charge densities and the eigenvalue densities. Thus, we have concluded that since the bubbling geometries are completely determined by the charge densities the geometries are constructed from the eigenvalue densities of the gauge theories with  $SU(2|4)$  symmetry.

Finally, let us comment on IIA LST on  $R \times S^5$ , which is another theory with  $SU(2|4)$  symmetry. Like other  $SU(2|4)$  symmetric theories, LST on  $R \times S^5$  is thought to have many discrete vacua and for each vacuum there exists a gravity dual given by type IIA bubbling geometry [13]. The gravity dual of LST around the trivial vacuum was elaborated in [16] and shown to be obtained from a double scaling limit of the gravity dual of PWMM around a particular vacuum. Although it is also expected that the gravity dual of LST around a general vacuum can be obtained from the same kind of double scaling limit of the gravity dual of PWMM, some careful analysis seems to be needed. So, we will return to this issue in a separate paper.

## Acknowledgments

The work of Y.A. and S.S. is supported in part by the JSPS Research Fellowship for Young Scientists. The work of S.S is also supported in part by the JSPS Grant-in-Aid for Scientific Research (A) No. 23244057.

## A Dual integral equations

Let  $A(u)$  be a function defined on  $[0, \infty)$ . We define two functions on  $[0, \infty)$  using  $A(u)$  as

$$\phi(x) = \int_0^\infty u^{-2\alpha} (1 + k(u)) A(u) J_\nu(xu) du, \quad (\text{A.1})$$

$$\chi(x) = \int_0^\infty A(u) J_\nu(xu) du, \quad (\text{A.2})$$

where  $\alpha$  is a positive half integer,  $k(u)$  is a given weight function on  $[0, \infty)$  and  $J_\nu(z)$  is the Bessel function of the first kind of order  $\nu$  defined by

$$J_\nu(z) = \sum_{s=0}^{\infty} \frac{(-1)^s (z/2)^{\nu+2s}}{s! \Gamma(\nu + s + 1)}. \quad (\text{A.3})$$

In this appendix, we consider a problem of finding a solution for  $A(u)$ , which solves the following equations called the dual integral equations.

$$\phi(x) = F(x) \quad \text{for } x \in I_1. \quad (\text{A.4})$$

$$\chi(x) = G(x) \quad \text{for } x \in I_2. \quad (\text{A.5})$$

Here we have divided the positive real line  $[0, \infty)$  to two segments denoted by  $I_1$  and  $I_2$ , which are written as  $[0, c)$  and  $[c, \infty)$ , respectively.  $F(x)$  and  $G(x)$  are assumed to be known functions defined on  $I_1$  and  $I_2$ , respectively. We will see that the problem reduces to a problem of solving a single Fredholm integral equation of the second kind [27].

In the following, for any function  $f(x)$  on  $[0, \infty)$ , we denote by  $f_1(x)$  and  $f_2(x)$  the restrictions of  $f(x)$  to  $I_1$  and  $I_2$ , respectively. We assume that  $f_i(x) = 0$  unless  $x \in I_i$  ( $i = 1, 2$ ), so that the original function can be written as  $f(x) = f_1(x) + f_2(x)$ . For example, the equation (A.4) can be written in this notation as  $\phi_1(x) = F(x)$  for  $x \in I_1$ .

We also introduce the modified Hankel transformation,

$$S_{\eta, \alpha} f(x) := \left(\frac{2}{x}\right)^\alpha \int_0^\infty t^{1-\alpha} f(t) J_{2\eta+\alpha}(xt) dt. \quad (\text{A.6})$$

The inverse transformation is given by

$$S_{\eta, \alpha}^{-1} = S_{\eta+\alpha, -\alpha}. \quad (\text{A.7})$$

It is easy to see that

$$\phi(x) = \left(\frac{x}{2}\right)^\alpha S_{\nu/2-\alpha, 2\alpha} \{(1+k) \cdot \psi\}(x), \quad (\text{A.8})$$

$$\chi(x) = S_{\nu/2, 0} \psi(x), \quad (\text{A.9})$$

where  $\psi(u)$  is defined by

$$A(u) = u\psi(u), \quad (\text{A.10})$$

and the dot in (A.8) denotes the product of functions defined as usual by  $f \cdot g(x) = f(x)g(x)$ .

We first put

$$\psi(u) = S_{\nu/2, -\alpha} h(u), \quad (\text{A.11})$$

and substitute this to (A.9). Then, after performing the inverse transformation (A.7) twice, we obtain

$$h(x) = S_{\nu/2+\alpha, -\alpha} S_{\nu/2, 0} \chi(x) = K_{\nu/2+\alpha, -\alpha} \chi(x), \quad (\text{A.12})$$

where we have defined

$$K_{\eta, \alpha+\beta} = S_{\eta, \alpha} S_{\eta+\alpha, \beta}. \quad (\text{A.13})$$

One can show that the transformation  $K_{\eta, \alpha}$  can be written as

$$K_{\eta, \alpha} f(x) = \begin{cases} \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_x^\infty (u^2 - x^2)^{\alpha-1} u^{-2\alpha-2\eta+1} f(u) du & \text{for } 0 < \alpha, \\ -\frac{x^{2\eta-1}}{\Gamma(1+\alpha)} \frac{d}{dx} \int_x^\infty u^{-2\alpha-2\eta+1} (u^2 - x^2)^\alpha f(u) du & \text{for } -1 < \alpha < 0. \end{cases} \quad (\text{A.14})$$

See [27] for the definition for  $\alpha < -1$ . From (A.12) and (A.14), we find that the solution for  $h_2(x)$  is given by

$$h_2(x) = K_{\nu/2+\alpha, -\alpha} \chi_2(x) = K_{\nu/2+\alpha, -\alpha} G(x). \quad (\text{A.15})$$

Similarly, by applying the same calculation to (A.8), one can obtain the following equation for  $x \in I_1$ .

$$h_1(x) + S_{\nu/2-\alpha, \alpha} \{k \cdot S_{\nu/2, -\alpha} h_1\}(x) = H(x). \quad (\text{A.16})$$

The function  $H(x)$  is defined by<sup>7</sup>

$$H(x) = \left(\frac{2}{x}\right)^{2\alpha} I_{\nu/2, -\alpha} F(x) - S_{\nu/2-\alpha, \alpha} k(x) S_{\nu/2, 0} G(x), \quad (\text{A.17})$$

where  $I_{\eta, \alpha+\beta}$  is defined by

$$I_{\eta, \alpha+\beta} = S_{\eta+\alpha, \beta} S_{\eta, \alpha}, \quad (\text{A.18})$$

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<sup>7</sup>The last term in (A.17) is obtained by using (A.15) and the relation,  $S_{\eta, \alpha} K_{\eta+\alpha, \beta} = S_{\eta, \alpha+\beta}$ .

and it is written more explicitly as

$$I_{\eta,\alpha}f(x) = \begin{cases} \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^x u^{2\eta+1}(x^2-u^2)^{\alpha-1}f(u)du & \text{for } 0 < \alpha, \\ \frac{x^{-2\alpha-2\eta-1}}{\Gamma(1+\alpha)} \frac{d}{dx} \int_0^x u^{2\eta+1}(x^2-u^2)^\alpha f(u)du & \text{for } -1 < \alpha < 0. \end{cases} \quad (\text{A.19})$$

Note that  $H(x)$  depends only on the known functions  $k(x)$ ,  $F(x)$  and  $G(x)$ . For the second term in the left-hand side of (A.16), we interchange the order of the integration, so that

$$S_{\nu/2-\alpha,\alpha}\{k \cdot S_{\nu/2,-\alpha}h_1\}(x) = \int_0^1 K(x,u)h_1(u)du, \quad (\text{A.20})$$

where the integral kernel  $K(x,u)$  is defined by

$$K(x,u) = u \left(\frac{u}{x}\right)^\alpha \int_0^\infty tk(t)J_{\nu-\alpha}(xt)J_{\nu-\alpha}(ut)dt. \quad (\text{A.21})$$

Hence, we conclude that  $h_1$  is the solution of the Fredholm integral equation of the second kind,

$$h_1(x) + \int_0^1 K(x,u)h_1(u)du = H(x). \quad (\text{A.22})$$

The two equations (A.15) and (A.22) fully determines  $h(x) = h_1(x) + h_2(x)$ .

Now, we consider a generalization of this result to vector-valued functions. This is easy since the above problem is linear in  $A(u)$ . Let us consider a set of functions  $\{A_s(u)|s = 1, \dots, n\}$  which are determined by the equations,

$$\phi_s(x) = F_s(x) \quad \text{for } x \in I_1^{(s)}, \quad (\text{A.23})$$

$$\chi_s(x) = G_s(x) \quad \text{for } x \in I_2^{(s)}, \quad (\text{A.24})$$

where  $\phi_s(x)$  and  $\chi_s(x)$  are now defined by

$$\phi_s(x) = \int_0^\infty u^{-2\alpha} \sum_{t=1}^n (\delta_{st} + k_{st}(u))A_t(u)J_\nu(xu)du, \quad (\text{A.25})$$

$$\chi_s(x) = \int_0^\infty A_s(u)J_\nu(xu)du. \quad (\text{A.26})$$

For each  $s$ ,  $F_s(x)$  and  $G_s(x)$  are assumed to be known functions and  $I_{1,2}^{(s)}$  are the two connected intervals, the sum of which is equal to  $[0, \infty)$ . If we write for each  $s$

$$A_s(u) = uS_{\nu/2,-\alpha}h_s(u), \quad (\text{A.27})$$

it is easy to see that  $h_s(u)$  are determined by the following equations.

$$h_{s2}(x) = K_{\nu/2+\alpha, -\alpha} G_s(x). \quad (\text{A.28})$$

$$h_{s1}(x) + \sum_{t=1}^n \int_0^1 K_{st}(x, u) h_{t1}(u) du = H_s(x), \quad (\text{A.29})$$

where  $H_s(x)$  and  $K_{st}(x, y)$  are defined by

$$H_s(x) = \left(\frac{2}{x}\right)^{2\alpha} I_{\nu/2, -\alpha} F_s(x) - \sum_{t=1}^n S_{\nu/2-\alpha, \alpha} k_{st}(x) S_{\nu/2, 0} G_t(x), \quad (\text{A.30})$$

and

$$K_{st}(x, u) = u \left(\frac{u}{x}\right)^\alpha \int_0^\infty t k_{st}(t) J_{\nu-\alpha}(xt) J_{\nu-\alpha}(ut) dt. \quad (\text{A.31})$$

## B SYM on $R \times S^2$ and SYM on $R \times S^3/Z_k$

In this appendix, we review SYM on  $R \times S^2$  and SYM on  $R \times S^3/Z_k$ . See for detail [18, 24].

### SYM on $R \times S^2$

The action of SYM on  $R \times S^2$  is given by

$$\begin{aligned} S_{R \times S^2} = \frac{1}{g_{S^2}^2} \int d\tau d\Omega_2 \text{Tr} & \left( -\frac{1}{2} (\partial_1 X_b - i L_b^{(0)} X_1 - i [X_1, X_b])^2 - \frac{1}{2} (D_1 X_m)^2 \right. \\ & - \frac{1}{4} (2\varepsilon_{abc} X_c + i L_a^{(0)} X_b - i L_b^{(0)} X_a - i [X_a, X_b])^2 \\ & + \frac{1}{2} (L_a^{(0)} X_m - [X_a, X_m])^2 - \frac{1}{2} X_m^2 + \frac{1}{4} [X_m, X_n]^2 \\ & \left. - \frac{i}{2} \Psi \Gamma^1 \partial_1 \Psi + \frac{1}{2} \Psi \Gamma^a L_a^{(0)} \Psi - \frac{3i}{8} \Psi \Gamma^{234} \Psi - \frac{1}{2} \Psi \Gamma^M [X_M, \Psi] \right), \end{aligned} \quad (\text{B.1})$$

where  $L_a^{(0)}$  are ordinary angular momentum operators and the radius of  $S^2$  is set to be  $\frac{1}{2}$  [18, 24].  $X_1$  is the gauge field corresponding to the  $R$  direction,  $X_m$  are the scalar fields and  $\Psi$  is a ten-dimensional Majorana-Weyl spinor with 16 components.  $X_a$  include gauge fields and a scalar field:

$$\vec{X} = \Phi \vec{e}_r + a_2 \vec{e}_\varphi - a_3 \vec{e}_\theta, \quad (\text{B.2})$$



where  $\vec{X} = (X_2, X_3, X_4)$ ,  $\vec{e}_r = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ ,  $\vec{e}_\theta = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta)$  and  $\vec{e}_\varphi = (-\sin \varphi, \cos \varphi, 0)$ .  $a_2$  and  $a_3$  are the gauge fields in the local Lorentz frame and  $\Phi$  is the scalar field. The vacuum of this theory is given by the Dirac monopole

$$\begin{aligned}\hat{a}_2 &= 0, \quad \hat{a}_3 = -\frac{\cos \theta \mp 1}{\sin \theta} \hat{\Phi}, \\ \hat{\Phi} &= 2 \operatorname{diag}(q_1 \mathbf{1}_{N_1}, \dots, q_s \mathbf{1}_{N_s}, \dots, q_\Lambda \mathbf{1}_{N_\Lambda}),\end{aligned}\tag{B.3}$$

where  $q_s \in Z/2$  are monopole charges. The upper and lower signs represent the northern region where  $0 \leq \theta < \pi$  and the southern region where  $0 < \theta \leq \pi$  on  $S^2$ , respectively.

The localization can be applied to the following scalar field sitting at  $(\tau, \theta, \varphi) = (\tau, 0, 0)$ ,

$$\phi(\tau, 0, 0) = -X_4(\tau, 0, 0) + \sinh \tau X_9(\tau, 0, 0) + i \cosh \tau X_{10}(\tau, 0, 0).\tag{B.4}$$

This scalar field preserves the same supersymmetries as (3.7). Note that  $X_4(\tau, 0, 0) = \Phi(\tau, 0, 0)$ . The vev of operators made of (B.4) can be computed in terms of the matrix integral obtained in the D2-brane limit of (3.10) with (3.11).

### **SYM on $R \times S^3/Z_k$**

$S^3/Z_k$  is the  $Z_k$ -orbifold of the round  $S^3$  along the  $S^1$ -fiber direction. When  $S^3$  is parametrized by the coordinates  $(\theta, \varphi, \psi)$  ( $0 \leq \theta \leq \pi$ ,  $0 \leq \varphi \leq 2\pi$ ,  $0 \leq \psi \leq 4\pi$ ) with the metric

$$ds_{S^3}^2 = \frac{1}{4} \{d\theta^2 + \sin^2 \theta d\varphi^2 + (d\psi + \cos \theta d\varphi)^2\},\tag{B.5}$$

$S^3/Z_k$  is realized as the following coordinate identification;  $(\theta, \varphi, \psi) \sim (\theta, \varphi, \psi + 4\pi/k)$ . Then, SYM on  $R \times S^3/Z_k$  is obtained by keeping the modes of fields of SYM on  $R \times S^3$  that are invariant under the  $Z_k$  action.

The action of SYM on  $R \times S^3/Z_k$  takes the form,

$$\begin{aligned}S_{R \times S^3/Z_k} &= \frac{1}{g_{S^3/Z_k}^2} \int d\tau d\Omega_{S^3/Z_k} \operatorname{Tr} \left[ -\frac{1}{2} (\partial_1 X_b - i \mathcal{L}_b X_1 - i [X_1, X_b])^2 - \frac{1}{2} (D_1 X_m)^2 \right. \\ &\quad \left. - \frac{1}{4} (2\varepsilon_{abc} X_c + i \mathcal{L}_a X_b - i \mathcal{L}_b X_a - i [X_a, X_b])^2 \right. \\ &\quad \left. + \frac{1}{2} (\mathcal{L}_a X_m - [X_a, X_m])^2 - \frac{1}{2} X_m^2 + \frac{1}{4} [X_m, X_n] [X^m, X^n] \right]\end{aligned}$$

$$-\frac{i}{2}\Psi\Gamma^1\partial_1\Psi + \frac{1}{2}\Psi\Gamma^a\mathcal{L}_a\Psi - \frac{3i}{8}\Psi\Gamma^{234}\Psi - \frac{1}{2}\Psi\Gamma^M[X_M, \Psi], \quad (\text{B.6})$$

where  $\mathcal{L}_a$  are the Killing vectors on  $S^3/Z_k$  [18, 24].  $X_1$  and  $X_a$  are the gauge fields for  $R$  and  $S^3/Z_k$  directions, respectively,  $X_m$  are the scalar fields and  $\Psi$  is a ten-dimensional Majorana-Weyl spinor with 16 components.

The vacuum of this theory is parametrized by the holonomy around the  $S^1$ -fiber direction of  $S^3/Z_k$ . The holonomy can be labeled by

$$\{(e^{2\pi i\alpha/k}, N_2^{(\alpha)})\}_{\alpha \in K}, \quad (\text{B.7})$$

where  $K \subset \{1, 2, \dots, k\}$  and  $N_2^{(\alpha)}$  are the multiplicities of the holonomy phase  $e^{2\pi i\alpha/k}$ . The total matrix size is given by  $N_2 = \sum_{\alpha \in K} N_2^{(\alpha)}$ . Note that when  $k = 1$  this theory is the SYM on  $R \times S^3$  and has the unique and trivial vacuum.

The localization can be applied to the following Wilson loop operator defined at a fixed  $\tau$ ,

$$W(\tau) = \frac{1}{N_2} \text{Tr} P \exp \left( -2\pi i \int_0^1 ds \{ -X_4(x(s)) + \sinh \tau X_9(x(s)) + i \cosh \tau X_{10}(x(s)) \} \right), \quad (\text{B.8})$$

where  $x^\mu(s) = (\tau, 0, 0, 4\pi s)$ . This also preserves the same supersymmetries as (3.7). The correlation functions of (B.8) can be computed in terms of the matrix integral obtained by applying the T-duality to (3.13).

## C Condition for large $S^5$ radius

In this appendix, we show that  $\lambda^{(s)} \gg D_s$  is a sufficient condition for the large  $S^5$  radius at the tips of the disks in the electrostatic problem. At the tip of a disk, the disk radius  $R$  and the radius  $R_{S^5}$  of  $S^5$  are related as [16]

$$R = \frac{R_{S^5}^2}{4} \quad (\text{C.1})$$

in the string unit,  $\alpha' = 1$ . Then, under the identification (3.27), the  $S^5$  radius is large if and only if  $x_m^{(s)} \gg 1$ . In the following, we show that  $x_m^{(s)} \gg 1$  if  $\lambda^{(s)} \gg D_s$ . We assume

that the index  $s$  labels the disks in the order of the  $z$ -coordinate, namely,  $D_{s-1} < D_s$  ( $s = 1, 2, \dots, \Lambda$ ).

First we divide the theory described by (3.23) into three parts. The first is the free part, the action of which is given by

$$S_1 = \sum_{s=1}^{\Lambda} \int dx \left( \frac{2D_s}{g^2} (x^2 - \mu_s) \rho^{(s)}(x) + \frac{\pi}{2} (\rho^{(s)}(x))^2 \right). \quad (\text{C.2})$$

The second is the self-interaction part given by

$$S_2 = -\frac{1}{2} \sum_{s=1}^{\Lambda} \int dx dy \frac{2D_s}{(2D_s)^2 + (x-y)^2} \rho^{(s)}(x) \rho^{(s)}(y). \quad (\text{C.3})$$

The third is the interaction between different  $s$  and  $t$ , defined by

$$S_3 = -\frac{1}{2} \sum_{s \neq t} \int dx dy \left[ \frac{D_s + D_t}{(D_s + D_t)^2 + (x-y)^2} - \frac{|D_s - D_t|}{(D_s - D_t)^2 + (x-y)^2} \right] \rho^{(s)}(x) \rho^{(t)}(y). \quad (\text{C.4})$$

The total theory is described by the sum of these. But for the moment, let us consider more generally the theory defined by  $S(\alpha, \beta) = S_1 + \alpha S_2 + \beta S_3$ , where  $\alpha$  and  $\beta$  are parameters. We start with the simplest free theory with  $\alpha = \beta = 0$ . In this case, the extents of the eigenvalues can be easily estimated as  $x_m^{(s)} \sim (\lambda^{(s)}/D_s)^{1/3}$ . This gives a typical length scale of the free theory. From (C.3) and (C.4), one can also read off the typical length scale of the interaction potentials. For the self-interaction, it is given by  $\Delta x \sim D_s$ , where  $\Delta x$  denotes the separation distance between two eigenvalues. For the interaction between different  $s$  and  $t$ , the scale (for a fixed  $s$ ) is equal or greater than  $D_s^{3/4}$ , namely,  $\Delta x \gtrsim D_s^{3/4}$ . The lower bound is saturated by the interaction between  $s$  and  $t = s \pm 1$ . Then, let us consider turning on the interactions to recover the theory with  $\alpha = \beta = 1$ . The typical scale  $(\lambda^{(s)}/D_s)^{1/3}$  of the free theory should be modified by the interactions, which have structures with the length scale equal or greater than  $D_s^{3/4}$  (Note that we always assume that  $D_s \gg 1$ ). The modified scale should be at least greater than  $\min((\lambda^{(s)}/D_s)^{1/3}, D_s^{3/4})$ , since there is nothing which provides a finer scale than these. If the modified scale is  $x_m^{(s)} \sim D_s^{3/4}$ , this is always large enough when  $D_s \gg 1$ . If the modified scale is  $x_m^{(s)} \sim (\lambda^{(s)}/D_s)^{1/3}$ , this is large if  $\lambda^{(s)} \gg D_s$ . Therefore, we conclude that if  $\lambda^{(s)} \gg D_s$ , the typical extents of the eigenvalues are always much greater than 1.

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